

On some results concerning the polygonal polynomials

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ABSTRACT. In this paper we define the n th polygonal polynomial $P_n(z) = (z-1)(z^2-1)\cdots(z^n-1)$ and we investigate recurrence relations and exact integral formulae for the coefficients of $P_n(z)$ and for those of the Mahonian polynomials $Q_n(z) = (z+1)(z^2+z+1)\cdots(z^{n-1}+\cdots+z+1)$. We also explore numerical properties of these coefficients, unraveling new meanings for old sequences and generating novel entries to the Online Encyclopedia of Integer Sequences (OEIS). Some open questions are also formulated.

1. INTRODUCTION

For a positive integer n , we define the n th *polygonal polynomial* by

$$(1.1) \quad P_n(z) = (z-1)(z^2-1)\cdots(z^n-1).$$

For each $k = 1, \dots, n$, the roots of $z^k - 1$ are the complex coordinates of the vertices of the regular k -gon centered in the origin and having 1 as a vertex. Consequently, the roots of $P_n(z)$ are the vertices, with repetitions, of the regular k -gons, $k = 1, \dots, n$, motivating the name of this polynomial. Clearly, $P_n(z)$ has degree $\frac{n(n+1)}{2}$ and integer coefficients. These polynomials are closely linked to Euler's famous pentagonal number theorem concerning the infinite expansion $(1-x)(1-x^2)(1-x^3)\cdots = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}$, where $|x| < 1$, and the exponents are called (generalised) pentagonal numbers [4].

The polynomials $P_n(z)$ are a special case of the general class of polynomials defined by

$$F_{m_1, \dots, m_n}^{z_1, \dots, z_n}(z) = \prod_{k=1}^n (z^{m_k} - z_k),$$

where n, m_1, m_2, \dots, m_n are positive integers and z_1, z_2, \dots, z_n are complex numbers with $|z_1| = |z_2| = \cdots = |z_n| = 1$, introduced and investigated by Andrica and Bagdasar in [3]. This general class also contains the cyclotomic polynomials.

Recall that the n th cyclotomic polynomial Φ_n is defined by

$$(1.2) \quad \Phi_n(z) = \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k, n)=1}} (z - \zeta_n^k),$$

where $\zeta_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ denotes the first primitive root of order n of the unity. It is well known that the following formula holds

$$(1.3) \quad z^n - 1 = \prod_{d|n} \Phi_d(z).$$

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Therefore, we obtain the decomposition of $P_n(z)$ as product of cyclotomic polynomials

$$(1.4) \quad P_n(z) = \prod_{k=1}^n \prod_{d|k} \Phi_d(z).$$

Because every cyclotomic polynomial is irreducible over \mathbb{Z} ([7], Theorem 1, page 195), it follows that P_n has exactly $\nu(n) = \sum_{k=1}^n \tau(k)$ factors irreducible over \mathbb{Z} , where $\tau(k)$ denotes the number of divisors of the positive integer k . This property gives a new interpretation for the integer sequence $\nu(n)$ (indexed as [A006218](#) in the Online Encyclopedia of Integer Sequences (OEIS) [12]), in terms of the polynomial P_n . The terms of $\nu(n)$ are

1, 3, 5, 8, 10, 14, 16, 20, 23, 27, 29, 35, 37, 41, 45, 50, 52, 58, 60, 66, 70, 74, 76, 84, 87, 91, 95, ...

The explicit computations for the magnitude of coefficients of cyclotomic polynomials may involve complicated formulations [8, 9], [14, p.258-259]. The integral formulae for the coefficients of the cyclotomic polynomials established by Andrica and Bagdasar [2], opened new perspectives in the study of these coefficients and related sequences.

It is easy to see that the direct link between the Gaussian polynomials defined by

$$\binom{m}{r}_z = \frac{(z^m - 1) \cdots (z^{m-r+1} - 1)}{(z - 1) \cdots (z^r - 1)},$$

and the polygonal polynomials is given by the formula

$$(1.5) \quad \binom{m}{r}_z = \frac{P_m(z)}{P_r(z)P_{m-r}(z)}.$$

While the polynomial $P_n(z)$ seems very simple, from many points of view it can be seen as the “father” of the cyclotomic polynomials $\Phi_d(z)$ (see formula (1.4) above), and it hides deep algebraic, arithmetic and combinatorial properties. The natural companion to $P_n(z)$ is the Mahonian polynomial $Q_n(z)$ defined in (2.16), with a key role in the theory of partitions. The aim of this paper is to explore the polynomials $P_n(z)$ and $Q_n(z)$ and their coefficients. The coefficients of $Q_n(z)$ are known as Mahonian numbers ([A008302](#) in [12]).

In Section 2 we investigate the coefficients of $P_n(z)$, for which we establish a recursive formula (subsection 2.1), we deduce an exact integral formula (subsection 2.2) and give a combinatorial interpretation (subsection 2.3). We then analyze the companion polynomial $Q_n(z)$ (subsection 2.4), for whose coefficients we give recurrence formulae (Theorem 2.3, Proposition 2.1). We also show by a direct computation that the polynomial $Q_n(z)$ is unimodal (Proposition 2.2), and we establish a link between the coefficients of polynomials $P_n(z)$ and $Q_n(z)$ (Theorem 2.4).

Section 3 is devoted to some integer sequences related to polynomials $P_n(z)$ and $Q_n(z)$. First, we examine the number of distinct complex roots of $P_n(z)$ (subsection 3.1). Then, we analyze the sequence of middle coefficients of $P_n(z)$ and deduce an integral formula, and of $Q_n(z)$ (Kendall-Mann numbers), and propose some conjectures (subsection 3.2). Furthermore, in subsection 3.3 we conjecture that every integer n can be a coefficient of some polynomial $P_m(z)$ (property known to hold for cyclotomic polynomials [15]), the result being confirmed numerically for the first 10^5 numbers. We also propose a new integer sequence, defined by the smallest value m , for which the integer $n \geq 1$ appears as a coefficient in $P_m(z)$ ([A301701](#) in [12]). Finally, in subsection 3.4 we discuss the number of positive, negative, and zero coefficients of $P_n(z)$, adding new entries to the OEIS, namely the sequences [A301703](#), [A301704](#), and [A301705](#).

2. FORMULAE AND PROPERTIES OF THE COEFFICIENTS

Recall that a polynomial $f(z) = a_0 + a_1z + \cdots + a_mz^m$ of degree m is called

- palindromic (or reciprocal) if $f(z) = z^m f\left(\frac{1}{z}\right)$, i.e., $a_j = a_{m-j}$, $j = 0, \dots, m$;
- antipalindromic (or antireciprocal) if $f(z) = -z^m f\left(\frac{1}{z}\right)$, i.e., $a_j = -a_{m-j}$;
- unimodal if the sequence of its coefficients is unimodal, i.e., there exists an integer t (called mode), such that $a_1 \leq a_2 \leq \cdots \leq a_t$ and $a_t \geq a_{t+1} \geq \cdots \geq a_m$.

The algebraic form of the polynomial $P_n(z)$ in (1.1) is

$$(2.6) \quad P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j.$$

An easy computation shows that

$$P_n(z) = (-1)^n z^{\frac{n(n+1)}{2}} P_n\left(\frac{1}{z}\right),$$

hence $P_n(z)$ is palindromic for n even, i.e., we have $c_j^{(n)} = c_{\frac{n(n+1)}{2}-j}^{(n)}$, $j = 0, \dots, \frac{n(n+1)}{2}$.

Also, it is antipalindromic for n odd, i.e., we have $c_j^{(n)} = -c_{\frac{n(n+1)}{2}-j}^{(n)}$, $j = 0, \dots, \frac{n(n+1)}{2}$.

2.1. The recurrence formula for the coefficients of $P_n(z)$. The coefficients of the polynomial $P_n(z)$ can be obtained recursively. Notice that

$$(2.7) \quad P_n(z) = \prod_{k=1}^n (z^k - 1) = P_{n-1}(z) (z^n - 1).$$

Using the coefficients of $P_n(z)$ and $P_{n-1}(z)$, one obtains

$$(2.8) \quad P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j = \left(\sum_{j=0}^{\frac{n(n-1)}{2}} c_j^{(n-1)} z^j \right) (z^n - 1).$$

This indicates the following formula:

$$(2.9) \quad c_j^{(n)} = \begin{cases} -c_j^{(n-1)} & \text{if } j \in \{0, \dots, n-1\}, \\ c_{j-n}^{(n-1)} - c_j^{(n-1)} & \text{if } j \in \left\{n, \dots, \frac{n(n-1)}{2}\right\}, \\ c_j^{(n-1)} & \text{if } j \in \left\{\frac{n(n-1)}{2} + 1, \dots, \frac{n(n+1)}{2}\right\}. \end{cases}$$

Using the recurrence (2.9), we obtain the numbers in Table 1.

The values of the coefficients $c_j^{(n)}$ correspond to $(-1)^n T(n, j)$ of [A231599](#) in OEIS.

2.2. The integral formula for the coefficients $c_j^{(n)}$. Writing the polynomial $P_n(z)$ in the algebraic form, one obtains

$$P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j = (-1)^n (1-z)(1-z^2) \cdots (1-z^n),$$

therefore

$$c_j^{(n)} + \sum_{k \neq j} c_k^{(n)} z^{k-j} = (-1)^n z^{-j} (1-z)(1-z^2) \cdots (1-z^n).$$

| | |
|-------------|--|
| $c_j^{(1)}$ | -1, 1 |
| $P_1(z)$ | $-1 + z$ |
| $c_j^{(2)}$ | 1, -1, -1, 1 |
| $P_2(z)$ | $1 - z - z^2 + z^3$ |
| $c_j^{(3)}$ | -1, 1, 1, 0, -1, -1, 1 |
| $P_3(z)$ | $-1 + z + z^2 - z^4 - z^5 + z^6$ |
| $c_j^{(4)}$ | 1, -1, -1, 0, 0, 2, 0, 0, -1, -1, 1 |
| $P_4(z)$ | $1 - z - z^2 + 2z^5 - z^8 - z^9 + z^{10}$ |
| $c_j^{(5)}$ | -1, 1, 1, 0, 0, -1, -1, -1, 1, 1, 0, 0, -1, -1, 1 |
| $P_5(z)$ | $-1 + z + z^2 - z^5 - z^6 - z^7 + z^8 + z^9 + z^{10} - z^{13} - z^{14} + z^{15}$ |
| $c_j^{(6)}$ | 1, -1, -1, 0, 0, 1, 0, 2, 0, -1, -1, -1, -1, 0, 2, 0, 1, 0, 0, -1, -1, 1 |
| $P_6(z)$ | $1 - z - z^2 + z^5 + 2z^7 - z^9 - z^{10} - z^{11} - z^{12} + 2z^{14} + z^{16} - z^{19} - z^{20} + z^{21}$ |
| $c_j^{(7)}$ | -1, 1, 1, 0, 0, -1, 0, -1, -1, 0, 1, 1, 2, 0, 0, 0, -2, -1, -1, 0, 1, 1, 0, 1, 0, 0, -1, -1, 1 |
| $P_7(z)$ | $-1 + z + z^2 - z^5 - z^7 - z^8 + z^{10} + z^{11} + 2z^{12} - 2z^{16} - z^{17} - z^{18} + z^{20} + z^{21} + z^{23} - z^{26} - z^{27} + z^{28}$ |

TABLE 1. Polynomials $P_n(z)$ and their coefficients for $n = 1, 2, 3, 4, 5, 6, 7$.

Letting $z = \cos 2t + i \sin 2t$, by the well-known de Moivre formula, we obtain

$$\begin{aligned}
c_j^{(n)} + \sum_{k \neq j} c_k^{(n)} z^{k-j} &= (-1)^n (\cos 2jt - i \sin 2jt) \prod_{s=1}^n (2 \sin^2 st - 2i \sin st \cos st) \\
&= 2^n i^n (\cos 2jt - i \sin 2jt) \prod_{s=1}^n (\cos st + i \sin st) \prod_{s=1}^n \sin st \\
&= 2^n \left[\cos \left(\frac{n(n+1) - 4j}{2} t + \frac{n\pi}{2} \right) + i \sin \left(\frac{n(n+1) - 4j}{2} t + \frac{n\pi}{2} \right) \right] \prod_{s=1}^n \sin st.
\end{aligned}$$

Integrating on the interval $[0, \pi]$ we get

$$\begin{aligned}
c_j^{(n)} &= \frac{2^n}{\pi} \int_0^\pi \cos \left(\frac{n(n+1) - 4j}{2} t + \frac{n\pi}{2} \right) \sin t \sin 2t \cdots \sin nt \, dt \\
(2.10) \quad &= \begin{cases} \frac{(-1)^{\frac{n}{2}} 2^n}{\pi} \int_0^\pi \cos \left(\frac{n(n+1) - 4j}{2} t \right) \sin t \sin 2t \cdots \sin nt \, dt & \text{if } n \text{ is even} \\ \frac{(-1)^{\frac{n+1}{2}} 2^n}{\pi} \int_0^\pi \sin \left(\frac{n(n+1) - 4j}{2} t \right) \sin t \sin 2t \cdots \sin nt \, dt & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

In addition, from the proof of the integral formula (2.10) it follows that

$$\int_0^\pi \sin \left(\frac{n(n+1) - 4j}{2} t + \frac{n\pi}{2} \right) \sin t \sin 2t \cdots \sin nt \, dt = 0.$$

2.3. The combinatorial interpretation of the coefficients $c_j^{(n)}$. The calculation of these polynomial coefficients involves tuples with fixed sum. Let s, k, n be positive integers. We denote by $\alpha(s, k, n)$ the number of integer s -tuples (i_1, \dots, i_s) with the properties

$$(2.11) \quad i_1 + i_2 + \cdots + i_s = k, \quad 1 \leq i_1 < i_2 < \cdots < i_s \leq n.$$

The link with the coefficients $c_j^{(n)}$ is given in the following theorem.

Theorem 2.1. *The following formula holds :*

$$(2.12) \quad c_k^{(n)} = (-1)^{n-1} (\alpha(1, k, n) - \alpha(2, k, n) + \alpha(3, k, n) - \dots).$$

Proof. The coefficient $c_k^{(n)}$ of z^k in the expansion $(z-1)(z^2-1)\dots(z^n-1)$ involves s distinct terms chosen from the set $\{1, \dots, n\}$ with the property that their sum is k , and also $n-s$ terms equal to (-1) , for $s = 0, \dots, n$. Explicitly, $c_k^{(n)}$ is given by the expression

$$c_k^{(n)} = (-1)^{n-1} \alpha(1, k, n) + (-1)^{n-2} \alpha(2, k, n) + (-1)^{n-3} \alpha(3, k, n) + \dots.$$

This ends the proof. \square

Clearly, $\alpha(s, k, n)$ is an increasing function with n . When n is large enough, the function is stationary to a value not depending on n , simply denoted by $\alpha(s, k)$.

The following result shows a link between the polynomial $P_s(z)$ and integers $\alpha(s, k)$.

Theorem 2.2. *If s is a positive integer, then for all $z \in \mathbb{C}$ such that $|z| < 1$ we have*

$$(2.13) \quad \frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)} = \lim_{n \rightarrow \infty} \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq n} z^{i_1 + i_2 + \dots + i_s} = \sum_{k=0}^{\infty} \alpha(s, k) z^k.$$

Proof. Clearly, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{1 \leq i_1 < \dots < i_s \leq n} z^{i_1 + \dots + i_s} &= \sum_{1 \leq i_1 < \dots < i_s} z^{i_1 + \dots + i_s} = \sum_{1 \leq i_1 < \dots < i_{s-1}} z^{i_1 + \dots + i_{s-1}} \cdot \frac{z^{i_{s-1}+1}}{1-z} \\ &= \frac{z}{1-z} \sum_{1 \leq i_1 < \dots < i_{s-1}} z^{i_1 + \dots + 2i_{s-1}} = \frac{z}{1-z} \sum_{1 \leq i_1 < \dots < i_{s-2}} z^{i_1 + \dots + i_{s-2}} \cdot \frac{z^{2(i_{s-2}+1)}}{1-z^2} \\ &= \frac{z^{1+2}}{(1-z)(1-z^2)} \sum_{1 \leq i_1 < \dots < i_{s-2}} z^{i_1 + \dots + 3i_{s-2}} = \dots = \frac{z^{1+\dots+(s-1)}}{(1-z)\dots(1-z^{s-1})} \sum_{1 \leq i_1} z^{si_1} \\ &= \frac{z^{\frac{s(s+1)}{2}}}{(1-z)\dots(1-z^s)} = \frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)}. \end{aligned}$$

\square

Notice also that by (2.13) it follows that

$$(-1)^s z^{\frac{s(s+1)}{2}} = P_s(z) \left(\sum_{k=0}^{\infty} \alpha(s, k) z^k \right) = \left(\sum_{j=0}^{\frac{s(s+1)}{2}} c_j^{(s)} z^j \right) \left(\sum_{k=0}^{\infty} \alpha(s, k) z^k \right),$$

hence we obtain the following result.

Corollary 2.1. *Considering that $c_j^{(s)} = 0$ for $j > \frac{s(s+1)}{2}$, we have*

$$(2.14) \quad \sum_{j=0}^n \alpha(s, n-j) c_j^{(s)} = (-1)^s \delta_{n, \frac{s(s+1)}{2}},$$

where $\delta_{u,v}$ denotes the Kronecker symbol.

Corollary 2.2. *Let $\beta(s, k)$ be the number of distinct solutions to the equation*

$$(2.15) \quad j_1 + 2j_2 + \dots + sj_s = k,$$

such that the numbers $j_1, j_2, \dots, j_s \geq 1$ are not necessarily distinct. We have $\alpha(s, k) = \beta(s, k)$.

Proof. Indeed, for $j \in \{1, \dots, s\}$ and $|z| < 1$ one has $\frac{z^j}{1-z^j} = z^j + z^{2j} + \dots$. By taking the product over $j \in \{1, \dots, s\}$, one obtains

$$\frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)} = (z + z^2 + \dots)(z^2 + z^4 + \dots) \dots (z^s + z^{2s} + \dots) = \sum_{k=0}^{\infty} \beta(s, k) z^k.$$

The result follows by (2.13). Clearly, $\alpha(s, k) = \beta(s, k) = 0$ for $0 \leq k \leq \frac{s(s+1)}{2} - 1$. \square

Recent results concerning the number of ordered partitions of the integer k into s parts each of size at least 0 but no larger than n have been obtained in [13] and [16].

2.4. The connection to the Mahonian polynomial $Q_n(z)$. The polygonal polynomial can also be written as $P_n(z) = (z-1)^n Q_n(z)$, where

$$(2.16) \quad Q_n(z) = (z+1)(z^2+z+1) \dots (z^{n-1} + z^{n-2} + \dots + z + 1) = \sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)} z^j.$$

We call $Q_n(z)$ as the Mahonian polynomial, which presents interest in its own right and has been investigated in many papers [11] (related to coin-tossing), or Margolius [10]. The coefficients $a_k^{(n)}$ are called Mahonian numbers, representing the number of permutations of the set $\{1, \dots, n\}$ with k inversions, indexed as [A008302](#) in OEIS. If $z^k = z^{k_1} \dots z^{k_{n-1}}$ with z^{k_j} coming from the factor $1 + z + \dots + z^j$ in (2.16), then a direct interpretation of the coefficient $a_k^{(n)}$ is the number of partitions of the integer $k = k_1 + \dots + k_{n-1}$, with the constraints $0 \leq k_j \leq j$, $1 \leq j \leq n-1$. These numbers are related to the Mahonian distribution, which interestingly, are used in the mixing of diffusing particles [5].

| | |
|-------------|---|
| $a_j^{(2)}$ | 1, 1 |
| $Q_2(z)$ | $1 + z$ |
| $a_j^{(3)}$ | 1, 2, 2, 1 |
| $Q_3(z)$ | $1 + 2z + 2z^2 + z^3$ |
| $a_j^{(4)}$ | 1, 3, 5, 6, 5, 3, 1 |
| $Q_4(z)$ | $1 + 3z + 5z^2 + 6z^3 + 5z^4 + 3z^5 + z^6$ |
| $a_j^{(5)}$ | 1, 4, 9, 15, 20, 22, 20, 15, 9, 4, 1 |
| $Q_5(z)$ | $1 + 4z + 9z^2 + 15z^3 + 20z^4 + 22z^5 + 20z^6 + 15z^8 + 9z^8 + 4z^9 + z^{10}$ |
| $a_j^{(6)}$ | 1, 5, 14, 29, 49, 71, 90, 101, 101, 90, 71, 49, 29, 14, 5, 1 |
| $Q_6(z)$ | $1 + 5z + 14z^2 + 29z^3 + 49z^4 + 71z^5 + 90z^6 + 101z^7 + 101z^8 + 90z^9 + 71z^{10} + 49z^{11} + 29z^{12} + 14z^{13} + 5z^{14} + z^{15}$ |

TABLE 2. Polynomials $Q_n(z)$ and their coefficients for $n = 2, 3, 4, 5, 6$.

For $n = 10$ the following formula is obtained for polynomial $Q_{10}(z)$:

$$\begin{aligned} Q_{10}(z) = & 1 + 9z + 44z^2 + 155z^3 + 440z^4 + 1068z^5 + 2298z^6 + 4489z^7 + 8095z^8 + 13640z^9 \\ & + 21670z^{10} + 32683z^{11} + 47043z^{12} + 64889z^{13} + 86054z^{14} + 110010z^{15} + 135853z^{16} \\ & + 162337z^{17} + 187959z^{18} + 211089z^{19} + 230131z^{20} + 243694z^{21} + 250749z^{22} + 250749z^{23} \\ & + 243694z^{24} + 230131z^{25} + 211089z^{26} + 187959z^{27} + 162337z^{28} + 135853z^{29} \\ & + 110010z^{30} + 86054z^{31} + 64889z^{32} + 47043z^{33} + 32683z^{34} + 21670z^{35} + 13640z^{36} \\ & + 8095z^{37} + 4489z^{38} + 2298z^{39} + 1068z^{40} + 440z^{41} + 155z^{42} + 44z^{43} + 9z^{44} + z^{45}. \end{aligned}$$

We also give a recurrence formula for $a_j^{(n)}$, as a function of the coefficients of $Q_j(z)$.

Theorem 2.3. *Let $n \geq k + 1$. The coefficient $a_k^{(n)}$ is given by the recursive formula*

$$(2.17) \quad a_k^{(n)} = \sum_{j=0}^k \binom{n-1-j}{k-j} a_j^{(k)}.$$

Proof. Notice that $a_k^{(n)}$ is the coefficient of z^k in $Q_k(z)(1+z+\dots+z^k)^{n-k}$. Clearly:

$$(1+z+\dots+z^k)^{n-k} = \left(\frac{1-z^{k+1}}{1-z} \right)^{n-k} = (1-z^{k+1})^{n-k} \left(\frac{1}{1-z} \right)^{n-k}.$$

For $|z| < 1$, the geometric series summation yields

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^s + \dots.$$

By differentiating $(n-k-1)$ times one obtains

$$(n-k-1)! \left(\frac{1}{1-z} \right)^{n-k} = \sum_{s=0}^{\infty} (s+1)(s+2)\dots(s+n-k-1)z^s,$$

hence

$$\left(\frac{1}{1-z} \right)^{n-k} = \sum_{s=0}^{\infty} \frac{(s+1)(s+2)\dots(s+n-k-1)}{(n-k-1)!} z^s = \sum_{s=0}^{\infty} \binom{s+n-1-k}{s} z^s.$$

The coefficient of z^k in

$$\left(\sum_{j=0}^{\frac{k(k-1)}{2}} a_j^{(k)} \right) (1-z^{k+1})^{n-k} \left(\frac{1}{1-z} \right)^{n-k},$$

is then given by

$$a_k^{(n)} = \sum_{s=0}^k a_{k-s}^{(k)} \binom{s+n-k-1}{s} = \sum_{j=0}^k a_j^{(k)} \binom{n-j-1}{k-j}.$$

□

Example. First, one may show that $a_0^{(n)} = 1$ and $a_1^{(n)} = n-1$. By Theorem 2.3 one obtains:

- $k = 2$: For $n \geq 3$ we have

$$a_2^{(n)} = \binom{n-1}{2} a_0^{(2)} + \binom{n-2}{1} a_1^{(2)} + \binom{n-3}{0} a_2^{(2)} = \frac{(n-2)(n+1)}{2};$$

- $k = 3$: For $n \geq 4$ we have

$$a_3^{(n)} = \binom{n-1}{3} a_0^{(3)} + \binom{n-2}{2} a_1^{(3)} + \binom{n-3}{1} a_2^{(3)} + \binom{n-4}{0} a_3^{(3)} = a_3^{(n)} = \frac{n(n^2-7)}{6};$$

- $k = 4$: For $n \geq 5$ we have

$$\begin{aligned} a_4^{(n)} &= \binom{n-1}{4} a_0^{(4)} + \binom{n-2}{3} a_1^{(4)} + \binom{n-3}{2} a_2^{(4)} + \binom{n-4}{1} a_3^{(4)} + \binom{n-5}{0} a_4^{(4)} \\ &= \frac{n(n+1)(n^2+n-14)}{24}; \end{aligned}$$

- $k = 5$: For $n \geq 6$ we have

$$a_5^{(n)} = \binom{n-1}{5} a_0^{(5)} + \binom{n-2}{4} a_1^{(5)} + \binom{n-3}{3} a_2^{(5)} + \binom{n-4}{2} a_3^{(5)} + \binom{n-5}{1} a_4^{(5)} + \binom{n-6}{0} a_5^{(5)} = \frac{1}{120} (n-1)(n+6)(n^3 - 9n - 20);$$

- $k = 6$: For $n \geq 7$ we have

$$a_6^{(n)} = \frac{1}{720} n (n^5 + 9n^4 - 5n^3 - 165n^2 - 356n + 516).$$

For $n = 10$ we have $a_5^{(10)} = 1068$ and $a_6^{(10)} = 2298$, confirming the values of $Q_{10}(z)$.

The coefficients of the polynomial Q_n can also be obtained recursively. One may write

$$(2.18) \quad Q_n(z) = Q_{n-1}(z) (z^{n-1} + \cdots + z + 1).$$

Using the coefficients of $Q_n(z)$ and $Q_{n-1}(z)$, one obtains

$$(2.19) \quad Q_n(z) = \sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)} z^j = \left(\sum_{j=0}^{\frac{(n-2)(n-1)}{2}} a_j^{(n-1)} z^j \right) (z^{n-1} + \cdots + z + 1).$$

Proposition 2.1. *The following formula holds:*

$$(2.20) \quad a_j^{(n)} = \begin{cases} a_j^{(n-1)} + \cdots + a_0^{(n-1)} & \text{if } j \in \{0, \dots, n-1\}, \\ a_j^{(n-1)} + \cdots + a_{j-(n-1)}^{(n-1)} & \text{if } j \in \{n, \dots, \frac{n(n-1)}{2}\}. \end{cases}$$

One can prove that $Q_n(z)$ is a Λ -polynomial, i.e., it is both palindromic and unimodal with nonnegative coefficients. This follows from a general result by Andrews [1], and the fact that $Q_n(z)$ is a product of the Λ -polynomials $z+1, z^2+z+1, \dots, z^{n-1}+\cdots+z+1$. This property can be seen in Table 1. We also present a direct proof of the unimodality $Q_n(z)$.

Proposition 2.2. *The polynomial $Q_n(z)$ is unimodal.*

Proof. Let us denote for convenience $m = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$ and $M = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$. We shall prove the result by induction. Assume that the polynomial $Q_n(z)$ is unimodal. Observe that

(a) if $m = \frac{n(n-1)}{4}$, then the sequence $\{a_j^{(n)}\}_{j=0}^{n(n-1)/2}$ has a single maximum value at $j = m$, and also $a_{m-j}^{(n)} = a_{m+j}^{(n)}$, $j = 0, \dots, m$, by the unimodality of $Q_n(z)$;

(b) if $m \neq \frac{n(n-1)}{4}$, then sequence $\{a_j^{(n)}\}_{j=0}^{n(n-1)/2}$ has maximum values at $j = m, m+1$, and also $a_{m-j}^{(n)} = a_{m+1+j}^{(n)}$, $j = 0, \dots, m$, by the unimodality of $Q_n(z)$.

Clearly, $Q_2(z) = 1 + z$ is unimodal. We show that the sequence $a_j^{(n+1)}$, $j = 0, \dots, M$, is increasing. First, by formula (2.20), if $0 \leq j \leq n-1$, then $a_{j+1}^{(n+1)} - a_j^{(n+1)} = a_{j+1}^{(n)} \geq 1$. Then, if $n \leq j \leq m-1$, one has $a_{j+1}^{(n+1)} - a_j^{(n+1)} = a_{j+1}^{(n)} - a_{j-n}^{(n)} > 0$, as $Q_n(z)$ is unimodal. Finally, we prove that the inequality $a_{j+1}^{(n+1)} - a_j^{(n+1)} > 0$ holds whenever $m \leq j \leq M-1$. Denote for convenience $p = j - m$, for $p \in \{0, \dots, M - m - 1\}$.

Case 1. If $n = 4k$, then $M - m = 2k$. For $p \in \{0, \dots, 2k-1\}$ (even n), one obtains

$$a_{j+1}^{(n)} - a_j^{(n)} = a_{m+p+1}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p + 1 < n - p$. Also, one may notice that for $p = 2k$, one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+2k+1}^{(n)} - a_{m-2k}^{(n)} = a_{m+2k+1}^{(n)} - a_{m+2k}^{(n)} = a_{M+1}^{(n)} - a_M^{(n)} < 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \dots, M$ is increasing, and has the single mode $j = M$.

Case 2. If $n = 4k + 1$, then $M - m = 2k$. For $p \in \{0, \dots, 2k - 1\}$ (odd n) we have

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+p}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p < n - p$. Also, for $p = 2k$ one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+2k}^{(n)} - a_{m-2k}^{(n)} = a_{M+1}^{(n)} - a_M^{(n)} = 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \dots, M$ is increasing, and has modes $j = M, M + 1$.

Case 3. If $n = 4k + 2$, then $M - m = 2k + 1$. For $p \in \{0, \dots, 2k\}$ (even n), one has

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+p+1}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p + 1 < n - p$. Also, one may notice that for $p = 2k + 1$, one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+2k+2}^{(n)} - a_{m-(2k+1)}^{(n)} = a_{m+2k+2}^{(n)} - a_{m+2k+1}^{(n)} = a_{M+1}^{(n)} - a_M^{(n)} < 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \dots, M$ is increasing, and has the single mode $j = M$.

Case 4. If $n = 4k + 3$, then $M - m = 2k + 2$. For $p \in \{0, \dots, 2k + 1\}$ (odd n) we have

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+p}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p < n - p$. Also, for $p = 2k + 2$ one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+2k+2}^{(n)} - a_{m-2k-2}^{(n)} = a_{M+1}^{(n)} - a_M^{(n)} = 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \dots, M$ is increasing, and has modes $j = M, M + 1$. \square

Using the definition of the polynomial $Q_n(z)$, we obtain another interpretation of the coefficients of polynomial $P_n(z)$, in terms of Kandall-Mann numbers. Indeed, from

$$P_n(z) = \left[z^n \binom{n}{0} - \binom{n}{1} z^{n-1} + \binom{n}{2} z^{n-2} - \dots + (-1)^n \binom{n}{n} \right] \left(\sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)} z^j \right),$$

one obtains a link between the coefficients of polynomial $P_n(z)$ and those of $Q_n(z)$.

Theorem 2.4. *The following formula holds*

$$c_j^{(n)} = \begin{cases} (-1)^n \left(a_j^{(n)} \binom{n}{0} - a_{j-1}^{(n)} \binom{n}{1} + \dots + (-1)^j a_0^{(n)} \binom{n}{j} \right) & \text{if } j \in \{0, \dots, n-1\}, \\ (-1)^n \left(a_j^{(n)} \binom{n}{0} - a_{j-1}^{(n)} \binom{n}{1} + \dots + (-1)^n a_{j-n}^{(n)} \binom{n}{n} \right) & \text{if } j \in \left\{ n, \dots, \frac{n(n-1)}{2} \right\}. \end{cases}$$

Example. Applying the formula in Theorem 2.4 for $j = 1, \dots, 6$ and $n \geq j + 1$ we obtain

$$\begin{aligned} c_0^{(n)} &= (-1)^n, \\ c_1^{(n)} &= (-1)^n (a_1^{(n)} - a_0^{(n)} \binom{n}{1}) = (-1)^n ((n-1) - n) = (-1)^{n+1}, \\ c_2^{(n)} &= (-1)^n (a_2^{(n)} - a_1^{(n)} \binom{n}{1} + a_0^{(n)} \binom{n}{2}) = (-1)^{n+1}, \\ c_3^{(n)} &= (-1)^n (a_3^{(n)} - a_2^{(n)} \binom{n}{1} + a_1^{(n)} \binom{n}{2} - a_0^{(n)} \binom{n}{3}) = 0, \\ c_4^{(n)} &= (-1)^n (a_4^{(n)} - a_3^{(n)} \binom{n}{1} + a_2^{(n)} \binom{n}{2} - a_1^{(n)} \binom{n}{3} + a_0^{(n)} \binom{n}{4}) = 0, \\ c_5^{(n)} &= (-1)^n (a_5^{(n)} - a_4^{(n)} \binom{n}{1} + a_3^{(n)} \binom{n}{2} - a_2^{(n)} \binom{n}{3} + a_1^{(n)} \binom{n}{4} - a_0^{(n)} \binom{n}{5}) = (-1)^n, \\ c_6^{(n)} &= (-1)^n (a_6^{(n)} - a_5^{(n)} \binom{n}{1} + a_4^{(n)} \binom{n}{2} - a_3^{(n)} \binom{n}{3} + a_2^{(n)} \binom{n}{4} - a_1^{(n)} \binom{n}{5} + a_0^{(n)} \binom{n}{6}) = 0. \end{aligned}$$

3. ASSOCIATED INTEGER SEQUENCES

3.1. The number of distinct roots. The number of distinct roots of $P_n(z)$ is given by

$$(3.21) \quad A(n) = \varphi(1) + \varphi(2) + \cdots + \varphi(n).$$

Indeed, by (1.1) one has $P_{n+1}(z) = P_n(z)(z^{n+1} - 1)$. The new roots added by $z^{n+1} - 1$ to the set of roots of $P_n(z)$, are those given by the primitive roots of order $n + 1$, whose number is $\varphi(n + 1)$. The result follows by induction.

The sequence $A(n)$ is indexed as [A002088](#) in the OEIS, starting with the values:

1, 2, 4, 6, 10, 12, 18, 22, 28, 32, 42, 46, 58, 64, 72, 80, 96, 102, 120, 128, 140, 150, 172, 180, 200, ...

The asymptotic formula for $A(n)$ is given in [4, Theorem 3.7, page 72]:

$$A(n) \sim \frac{3n^2}{\pi^2} + O(n \log n).$$

Proposition 3.3. *Let $1 \leq k \leq n$ be an integer. If $z_{p,k} = e^{2\pi i \frac{p}{k}}$ is a k^{th} primitive root, then the multiplicity of root $z_{p,k}$ in the polynomial $P_n(z)$ is $\left\lfloor \frac{n}{k} \right\rfloor$. Consequently, one recovers the identity*

$$\sum_{k=1}^n \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

Proof. The root $z_{p,k}$ appears for the first time in polynomial P_k from the factor $z^k - 1$. Each of the $\varphi(k)$ roots appears as a non-primitive root of every multiple of k smaller than n . \square

3.2. The middle coefficients of $P_n(z)$ and $Q_n(z)$. Formula (2.10) provides an insight into the sequence of middle terms. Denote by $m = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$.

- If $n = 4k$, then for $m = \frac{n(n+1)}{4}$ we have $\cos\left(\frac{n(n+1)-4m}{2}t\right) = 1$ and

$$(3.22) \quad c_m^{(n)} = \frac{2^n}{\pi} \int_0^\pi \sin t \sin 2t \cdots \sin nt \, dt.$$

This sequence recovers [A269298](#) in OEIS, having the starting values

2, 2, 4, 6, 8, 16, 28, 50, 100, 196, 388, 786, 1600, 3280, 6780, 14060, 29280, 61232, ...

- If $n = 4k + 1$, then $n(n+1) - 4m = 2$ and $\sin\left(\frac{n(n+1)-4m}{2}t\right) = \sin t$, hence

$$(3.23) \quad c_m^{(n)} = -\frac{2^n}{\pi} \int_0^\pi \sin^2 t \sin 2t \cdots \sin nt \, dt = -c_{m+1}^{(n)}.$$

The sequence of terms multiplied by (-1) is not currently indexed in OEIS:

1, 1, 1, 1, 2, 2, 3, 4, 6, 10, 17, 28, 52, 94, 176, 339, 651, 1268, 2505, 4965, 9916, 19926, ...

- If $n = 4k + 2$, then $n(n+1) - 4m = 2$ and $\cos\left(\frac{n(n+1)-4m}{2}t\right) = \cos t$, hence

$$(3.24) \quad c_m^{(n)} = -\frac{2^n}{\pi} \int_0^\pi \cos t \sin t \sin 2t \cdots \sin nt \, dt = c_{m+1}^{(n)}.$$

The sequence of terms multiplied by (-1) is not currently indexed in OEIS:

1, 1, 2, 3, 5, 10, 19, 34, 68, 135, 269, 544, 1111, 2274, 4694, 9729, 20237, 42260, 88538, ...

- If $n = 4k + 3$ then for $m = \frac{n(n+1)}{4}$ we have $\sin\left(\frac{n(n+1)-4m}{2}t\right) = 0$ and $c_m^{(n)} = 0$.

Conjecture 1. Let n be an integer and $m = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$. Then the middle coefficient $c_m^{(n)}$ is

- positive for $n = 4k$ (mentioned in [A231599](#), without a proof);
- negative for $n = 4k + 1$ and $n = 4k + 2$.

The sequence of middle coefficients of $Q_n(z)$ gives the Kendall-Mann numbers indexed as [A000140](#) in OEIS, representing the number of permutations of the set $\{1, \dots, n\}$ having the maximum number of inversions. The first terms are:

1, 1, 2, 6, 22, 101, 573, 3836, 29228, 250749, 2409581, 25598186, 296643390, 3727542188, ...

The asymptotic behavior of this sequence was conjectured in [A000140](#).

Conjecture 2. Let n be an integer and $m = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$. Then the sequence of middle coefficients $a_m^{(n)}$ of $Q_n(z)$ satisfies the asymptotic formula

$$a_m^{(n)} \sim \frac{6n^{n-1}}{e^n}.$$

We are not aware of the existence of any proof at the moment.

For the cyclotomic polynomial $\Phi_n(z)$, the sequence of middle coefficients was studied by Dresden [6], and corresponds to the sequence [A094754](#) in OEIS.

3.3. First occurrence of n as a coefficient of a polygonal polynomial. It was proved by Suzuki [15], that every integer n is a coefficient for some cyclotomic polynomial $\Phi_m(z)$. We formulate the following open question regarding polygonal polynomials.

Conjecture 3. Every integer n appears as a coefficient of some polygonal polynomial.

Notice that 1 is a coefficient of $P_1(z)$, while 2 first appears as a coefficient in $P_4(z)$. For an integer $n \geq 0$, the sequence $a(n)$ defined by the smallest number m for which n is a coefficient of $P_m(z)$ produces sequence [A301701](#), recently added by the authors to OEIS.

| | | | | | | | | | | | | | | | | | | | | |
|--------|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|--------|
| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 ... |
| $a(n)$ | 3 | 1 | 4 | 10 | 12 | 17 | 16 | 19 | 20 | 22 | 22 | 23 | 24 | 25 | 25 | 25 | 24 | 26 | 26 | 28 ... |

The conjecture was recently checked for the first 10^5 integers.

3.4. Number of non-zero terms of $P_n(z)$. The sequence of non-zero coefficients of $P_n(z)$ is indexed as [A086781](#) in OEIS and starts with the terms

1, 2, 4, 6, 7, 12, 14, 18, 25, 32, 36, 42, 53, 68, 64, 84, 97, 108, 126, 146, 161, 170 ...

Recently, we have added the sequences below to the OEIS.

- [A301703](#), representing the number of positive coefficients of $P_n(z)$:

1, 2, 3, 3, 6, 6, 9, 13, 16, 18, 21, 27, 34, 32, 42, 47, 54, 62, 73, 79, 85, 96, 104, 113, 123, 140, 150, ...

- [A301704](#), representing the number of negative coefficients of $P_n(z)$:

1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 21, 26, 34, 32, 42, 50, 54, 64, 73, 82, 85, 96, 104, 116, 123, 134, 150, ...

- [A301705](#), representing the number of zero coefficients of $P_n(z)$:

0, 0, 1, 4, 4, 8, 11, 12, 14, 20, 25, 26, 24, 42, 37, 40, 46, 46, 45, 50, 62, 62, 69, 72, 80, 78, 79, 74, ...

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